Introduction to Symmetric functions and Representation theory

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2015.2

1. Symmetric functions

2. Combinatorics of Young tableaux

3. Representations of symmetric groups

4. Representations of general linear groups

1. Symmetric functions

- References

[1] I. G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, 2nd ed., 1995. (Chapter I)

[2] R. Stanley, *Enumerative Combinatorics*, Vol 2, Cambridge University Press, 1999. (Chapter 7)

Goal:

to introduce the notion of the ring of symmetric functions and some of its important linear bases

1.1 Ring of symmetric functions

- $x = (x_1, x_2, \ldots)$: a set of indeterminates

  $\alpha = (\alpha_1, \alpha_2, \ldots)$: a seq. of non-neg. integers with a finite sum

- A homogeneous symmetric function of degree n $(n \in \mathbb{N})$

  is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha \qquad (x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots)$$

(1) $\sum_i \alpha_i = n$

(2) $c_\alpha \in \mathbb{Z}$

(3) $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(1)}, \ldots)$ for all permutation σ of $\mathbb{N}$.

- Λ^n : the set of all symmetric functions of degree n ($\Lambda^0 = \mathbb{Z}$)

  $\Lambda := \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^2 \oplus \cdots$: the ring of symmetric functions.

  (graded ring, i.e. for $f \in \Lambda^m$, $g \in \Lambda^n$, we have $fg \in \Lambda^{m+n}$)

- $\Lambda_R = R \otimes_{\mathbb{Z}} \Lambda$ for $\mathbb{Z} \subset R$, (e.g. $\Lambda_{\mathbb{Q}}$, $\Lambda_{\mathbb{Q}[t],\ldots}$)

- $\Lambda_k^n := \Lambda^n|_{x_{k+1}=x_{k+2}=\cdots=0}$

  : the set of symmetric polynomials in k variables of degree n

  $\Lambda_k := \bigoplus_{n\geq 0} \Lambda_k^n = \mathbb{Z}[x_1, \ldots, x_k]^{\mathfrak{S}_k}$

  : the ring of symmetric polynomials in k variables

1.2 Partitions

- A partition of n is a sequence of non-negative integers

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \qquad \text{with } |\lambda| = \sum_k \lambda_k = n$$

$$\lambda = (4,2,1) \longleftrightarrow \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \qquad \text{Young diagram of } \lambda$$

$\ell(\lambda)$: the length of λ (the $\sharp$ of non-zero parts in λ)

- $\mathscr{P}$: the set of all partitions

- (Dominance ordering on $\mathscr{P}$) For $\lambda, \mu \in \mathscr{P}$,

$$\lambda \geq \mu \iff \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \text{ for all } k$$

$*$ $\geq$: not a linear ordering

1.3 Bases of Λ

- (Monomial symmetric functions)

$$m_\lambda(x) := \sum_\alpha x^\alpha \qquad (\lambda \in \mathscr{P})$$

where the sum ranges over all distinct permutations α of λ

e.g. $m_{(2)}(x) = \sum_i x_i^2, \quad m_{(2,1)}(x) = \sum_{i \neq j} x_i^2 x_j$

- $\{\, m_\lambda(x) \,|\, |\lambda| = n \,\}$: a $\mathbb{Z}$-basis of Λ^n

- $\{\, m_\lambda(x) \,|\, \lambda \in \mathscr{P} \,\}$: a $\mathbb{Z}$-basis of Λ

- (Elementary symmetric functions)

$$e_r(x) := \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} \quad (r \geq 0)$$

In other words, $E(t) = \prod_{i \geq 1}(1 + x_i t) = \sum_{r \geq 0} e_r(x) t^r$

$$e_\lambda(x) := e_{\lambda_1}(x) e_{\lambda_2}(x) \cdots \quad (\lambda \in \mathscr{P})$$

- $\{\, e_\lambda(x) \,|\, \lambda \in \mathscr{P} \,\}$: a $\mathbb{Z}$-basis of Λ

⋇ $\Lambda = \mathbb{Z}[e_1, e_2, \ldots]$ and $\{\, e_r(x) \,|\, r \geq 1 \,\}$ is algebraically independent over $\mathbb{Z}$

• (Complete symmetric functions)

$$h_r(x) := \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r} \quad (r \geq 0)$$

In other words, $H(t) = 1/\prod_{i \geq 1}(1 - x_i t) = \sum_{r \geq 0} h_r(x) t^r$

$$h_\lambda(x) := h_{\lambda_1}(x) h_{\lambda_2}(x) \cdots \quad (\lambda \in \mathscr{P})$$

- $\{\, h_\lambda(x) \mid \lambda \in \mathscr{P} \,\}$: a $\mathbb{Z}$-basis of Λ

※ $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ and $\{\, h_r(x) \mid r \geq 1 \,\}$ is algebraically independent over $\mathbb{Z}$

For $r \geq 1$, we have $\sum_{k=0}^{r} (-1)^k e_k(x) h_{r-k}(x) = 0$ (✍1)

• (Power sum symmetric functions)

$$p_r(x) := \sum_i x_i^r \quad (r \geq 1)$$

In other words, $P(t) = \frac{d}{dt} \log H(t) = \sum_{r \geq 1} p_r(x) t^{r-1}$

$$p_\lambda(x) := p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots \quad (\lambda \in \mathscr{P})$$

- $\{ p_\lambda(x) \mid \lambda \in \mathscr{P} \}$: a basis of $\Lambda_{\mathbb{Q}}$

⋇ $\Lambda_{\mathbb{Q}} = \mathbb{Q}[p_1, p_2, \ldots]$ and

$\{ p_r(x) \mid r \geq 1 \}$ is algebraically independent over $\mathbb{Q}$

e.g. $e_2(x) = \sum_{i<j} x_i x_j = \frac{1}{2}(p_1^2(x) - p_2(x))$.

• (Schur functions) $\lambda \in \mathscr{P}$ with $\ell(\lambda) \le n$

$$s_\lambda(x_1, \ldots, x_n) := \frac{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\sigma(x^{\lambda+\rho})}{\sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)\sigma(x^{\rho})}$$

where $\rho = (n-1, n-2, \ldots, 1, 0)$ (Schur **poly.** in n variables)

- $s_\lambda(x_1, \ldots, x_{n+1})|_{x_{n+1}=0} = s_\lambda(x_1, \ldots, x_n)$ $(n \ge 1)$

  $\exists!\ s_\lambda(x) \in \Lambda$ s.t. $s_\lambda(x)|_{x_{n+1}=x_{n+2}=\cdots=0} = s_\lambda(x_1, \ldots, x_n)$

  (called Schur function)

- $\{ s_\lambda(x) \,|\, \lambda \in \mathscr{P} \}$: a $\mathbb{Z}$-basis of Λ

1.4 Inner product on Λ

- Define a bilinear form $\langle\ ,\ \rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ by

$$\langle h_\lambda(x), m_\mu(x) \rangle := \delta_{\lambda\,\mu} \qquad (\lambda, \mu \in \mathscr{P})$$

Then we have

$$\langle p_\lambda(x), p_\mu(x) \rangle = z_\lambda \delta_{\lambda\,\mu}$$

$$\langle s_\lambda(x), s_\mu(x) \rangle = \delta_{\lambda\,\mu}$$

$*$ $z_\lambda = \prod_i i^{m_i}(m_i)!$ $\quad$ (m_i : $\sharp$ of parts equal to i in λ). $\qquad n!/z_\lambda =$???

- $\langle\ ,\ \rangle$ is symmetric and positive-definite
- $\{\, s_\lambda(x) \,|\, \lambda \in \mathscr{P} \,\}$: an orthonormal basis of Λ

$\because$ Let

$$\Pi(x,y)=\frac{1}{\prod_{i,j\geq 1}(1-x_iy_j)}.$$

(Lemma) $\{\,u_\lambda(x)\,\}$, $\{\,v_\mu(x)\,\}$: two bases of $\Lambda_{\mathbb{Q}}$. TFAE (✍2)

(1) $\Pi(x,y)=\sum_{\lambda\in\mathscr{P}}u_\lambda(x)v_\lambda(y)$

(2) $\langle u_\lambda(x),v_\mu(x)\rangle=\delta_{\lambda\,\mu}$

Also, we have

$$\Pi(x,y)=\sum_{\lambda\in\mathscr{P}}h_\lambda(x)m_\lambda(y)=\sum_{\lambda\in\mathscr{P}}z_\lambda^{-1}p_\lambda(x)p_\lambda(y) \qquad (✍3)$$

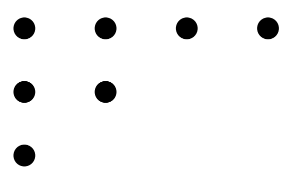
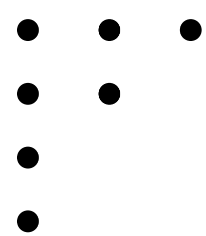
$$=\sum_{\lambda\in\mathscr{P}}s_\lambda(x)s_\lambda(y) \qquad \text{(Cauchy identity)}$$

- $\{\,s_\lambda(x)\,|\,\lambda\in\mathscr{P}\,\}$ is the unique $\mathbb{Z}$-basis diagonalizing $\Pi(x,y)$

1.5 More on Schur functions

- For simplicity, write $u = u(x)$ for a symmetric function $u(x)$
- Jacobi-Trudi formula

$$
\begin{aligned}
s_\lambda &= \det(h_{\lambda_i - i + j})_{1 \le i,j \le n}, \\
&= \det(e_{\lambda'_i - i + j})_{1 \le i,j \le n}, \qquad (\ \ell(\lambda), \ell(\lambda') \le n\ )
\end{aligned}
$$

$\lambda =$  $\lambda' =$  (conjugate of λ)

- $\lambda \in \mathscr{P}$

$$\lambda = (5,4,1,1) \longleftrightarrow \begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & & & & \\ \bullet & & & & \end{array} \qquad \text{(Young diagram)}$$

T: a tableau of shape λ

$$T = \begin{array}{ccccc} 1 & 2 & 4 & 3 & 2 \\ 5 & 6 & 2 & 1 & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$\mathrm{wt}(T) = (2,3,2,2,1,1)$: the weight of T

- T : standard $\Leftrightarrow$ the entries are $<$ (rows) and $\wedge$ (columns)

$$T = \begin{array}{lllll} 1 & 3 & 6 & 7 & 9 \\ 2 & 5 & 8 & 10 & \\ 4 & & & & \\ 11 & & & & \end{array}$$

T : semistandard $\Leftrightarrow$ the entries are $\leq$ (rows) and $\wedge$ (columns)

$$T = \begin{array}{lllll} 1 & 1 & 2 & 3 & 3 \\ 3 & 3 & 4 & 5 & \\ 4 & & & & \\ 5 & & & & \end{array}$$

• $\mathsf{ST}_n(\lambda)$: the set of all standard tableaux of shape λ with the entries from $\{1 < \cdots < n\}$

- $\mathsf{ST}_n(\lambda) \neq \emptyset$ if and only if $|\lambda| = n$

• $\mathsf{SST}_n(\lambda)$: the set of all semistandard tableaux of shape λ with the entries from $\{1 < \cdots < n\}$

- $\mathsf{SST}_n(\lambda) \neq \emptyset$ if and only if $\ell(\lambda) \leq n$

• Then we have for $\lambda \in \mathscr{P}$ with $\ell(\lambda) \leq n$

$$s_\lambda(x_1, \cdots, x_n) = \sum_{T \in \mathsf{SST}_n(\lambda)} x^{\mathsf{wt}(T)}$$

i.e. $s_\lambda(x_1, \cdots, x_n)$ is a weight generating function of $\mathsf{SST}_n(\lambda)$

- Kostka numbers $K_{\lambda\,\mu}$

$$s_\lambda = m_\lambda + \sum_{\lambda > \mu} K_{\lambda\,\mu} m_\mu$$

$K_{\lambda\mu}$ = the $\sharp$ of SST of shape λ and weight μ

- $>_R$: reverse lexicographical ordering on $\mathscr{P}$

$\lambda >_R \mu$ if and only if the first non-zero $\lambda_i - \mu_i$ is positive

((1) $>_R$: a linear ordering, (2) $\lambda \geq \mu \Longrightarrow \lambda \geq_R \mu$)

$6 >_R 51 >_R 42 >_R 411 >_R 33 >_R 321 >_R 3111 >_R 222$

$>_R 2211 >_R 2111 >_R 11111$

- $(K_{\lambda\,\mu})$: upper unitriangular matrix (with respect to $<_R$)

- $\{\, s_\lambda \,|\, \lambda \in \mathscr{P} \,\}$ can be obtained from $\{\, m_\lambda \,|\, \lambda \in \mathscr{P} \,\}$

  by Gram-Schmidt process;

$$\because \quad s_\lambda = m_\lambda + \sum_{\lambda > \mu} c_{\lambda\,\mu} m_\mu \quad \text{for some } c_{\lambda\mu}$$

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\,\mu}$$

- The action of the symmetric group $\mathfrak{S}_n$ on $\mathrm{SST}_n(\lambda)$

$$\begin{array}{ccccc} 1 & 1 & 2 & 3 & 3 \\ 3 & 3 & 4 & 5 & \\ 4 & & & & \\ 5 & & & & \end{array} \quad \overset{(3\ 4)}{\longleftrightarrow} \quad \begin{array}{ccccc} 1 & 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 5 & \\ 4 & & & & \\ 5 & & & & \end{array}$$

1.6 Graded Hopf algebra Λ

- Multiplication (Littlewood-Richardson coefficient)

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$$

- Comultiplication

  Define $\Delta : \Lambda \longrightarrow \Lambda \otimes \Lambda$ by

$$\Delta(e_r) := \sum_{p+q=r} e_p \otimes e_q \in \Lambda \otimes \Lambda$$

  Δ : coassociative

$$\begin{array}{ccccc} & \Lambda & \xrightarrow{\Delta} & \Lambda \otimes \Lambda & \\ \Delta & \downarrow & \circlearrowleft & \downarrow & \mathrm{id} \otimes \Delta \\ & \Lambda \otimes \Lambda & \xrightarrow[\Delta \otimes \mathrm{id}]{} & \Lambda \otimes \Lambda \otimes \Lambda & \end{array}$$

- 1 : a unit in Λ

  Counit $\varepsilon : \Lambda \longrightarrow \mathbb{Z}$ given by $\varepsilon|_{\Lambda^n} = 0$ $(n \geq 1)$ and $\varepsilon(1) = 1$

- $(\Lambda, m, \Delta, 1, \varepsilon)$: a graded Hopf algebra (over $\mathbb{Z}$)

- Λ : self-dual (i.e. $\Lambda^\circ = \bigoplus_{n \geq 0} (\Lambda^n)^* \simeq \Lambda$)

$\because \quad \Delta(s_\lambda) = \sum_{\mu,\nu} c^\lambda_{\mu\nu} s_\mu \otimes s_\nu$

$\Delta^*(s^*_\mu \otimes s^*_\nu)(s_\lambda) := (s^*_\mu \otimes s^*_\nu)(\Delta(s_\lambda)) = c^\lambda_{\mu\nu}$

$\therefore \quad \Delta^*(s^*_\mu \otimes s^*_\nu) = \sum_\lambda c^\lambda_{\mu\nu} s^*_\lambda$