Introduction to Symmetric functions and Representation theory

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2. Combinatorics of Young tableaux

- References

[1] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, 35. Cambridge University Press, 1997. (Part I)

[2] R. Stanley, *Enumerative Combinatorics*, Vol 2, Cambridge University Press, 1999. (Chapter 7)

Goal:

to describe multiplications of symmetric functions in terms of combinatorics of tableaux

2.1 Schensted's bumping rule

- $\lambda = (5,4,1,1) \longleftrightarrow$

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & & & & \\ \bullet & & & & \end{array}$$

(Young diagram)

- $\mathsf{SST}_n(\lambda)$: the set of Young tableaux of shape λ $\quad (\ell(\lambda) \leq n)$

  with the entries from $\{1 < \cdots < n\}$ $\quad (n \in \mathbb{N})$

$$T = \begin{array}{ccccc} 1 & 1 & 2 & 3 & 3 \\ 3 & 3 & 4 & 5 & \\ 4 & & & & \\ 5 & & & & \end{array} \qquad \mathrm{wt}(T) = (2,1,4,2,2)$$

- $s_\lambda(x_1, \cdots, x_n) = \sum_{T \in \mathsf{SST}_n(\lambda)} x^{\mathrm{wt}(T)}$

- Bumping algorithm $a \to T$

For $a \in \{1, \ldots, n\}$, $T \in \mathsf{SST}_n(\lambda)$, define $a \to T$ as follows;

(i) Let a' be the smallest entry in the first (or the left-most) column such that $a \leq a'$.

(ii) Replace a' by a. If there is no such a', put a at the bottom of the first column and stop the procedure.

(iii) Repeat (i) and (ii) on the next column with a' as far as possible.

(iv) Denote by $a \to T$ the resulting tableau.

e.g.

$$2 \quad \rightarrow \quad \begin{array}{cccccc} 1 & 1 & 2 & 5 & 6 \\ 2 & 3 & 4 & 6 & \\ 3 & & & & \\ 5 & & & & \end{array} \quad = \quad \begin{array}{cccccc} 1 & 1 & 2 & 4 & 5 & 6 \\ 2 & 2 & 3 & 6 & & \\ 3 & & & & & \\ 5 & & & & & \end{array}$$

- $\mathrm{sh}(T) \subset \mathrm{sh}(a \to T)$, $\quad |\mathrm{sh}(a \to T)/\mathrm{sh}(T)| = 1$

  ($\mathrm{sh}(T)$: the shape of T)

- The bumping algorithm is reversible

  if we fix a corner of a Young diagram

- Theorem

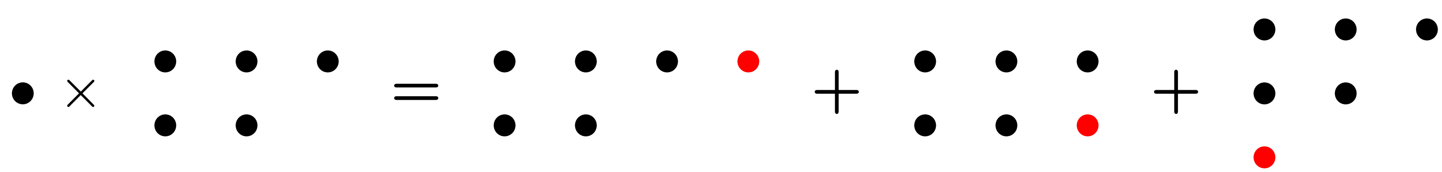
We have a weight preserving bijection

$$\mathsf{SST}_n(1) \times \mathsf{SST}_n(\lambda) \longrightarrow \bigsqcup_{\mu \supset \lambda,\ |\mu/\lambda|=1} \mathsf{SST}_n(\mu)$$

$$(a, T) \longmapsto (a \to T)$$

In terms of characters,

$$s_{(1)} \cdot s_\lambda = \sum_{|\mu/\lambda|=1} s_\mu$$



- $\mathbf{a} = a_1 \ldots a_k$: a word of length k with letters in $\{1, \ldots, n\}$

$(a_k \to \cdots (a_3 \to (a_2 \to a_1))) \in \mathsf{SST}_n(\lambda)$ for some λ with $|\lambda| = k$

e.g. $\mathbf{a} = 31224 = 3_1 1_2 2_3 2_4 4_5$

$$3 \rightsquigarrow 13 \rightsquigarrow \begin{array}{ll} 1 & 3 \\ \mathbf{2} & \end{array} \rightsquigarrow \begin{array}{lll} 1 & 2 & 3 \\ \mathbf{2} & & \end{array} \rightsquigarrow \begin{array}{lll} 1 & 2 & 3 \\ 2 & & \\ \mathbf{4} & & \end{array} =: P(\mathbf{a})$$

$$1 \rightsquigarrow 1\,2 \rightsquigarrow \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \rightsquigarrow \begin{array}{lll} 1 & 2 & 4 \\ 3 & & \end{array} \rightsquigarrow \begin{array}{lll} 1 & 2 & 4 \\ 3 & & \\ 5 & & \end{array} =: Q(\mathbf{a})$$

• Theorem (Robinson-Schensted correspondence)

We have a weight preserving bijection

$$\mathsf{SST}_n(1) \times \cdots \times \mathsf{SST}_n(1) \longrightarrow \bigsqcup_{|\lambda|=k} \mathsf{SST}_n(\lambda) \times \mathsf{ST}_k(\lambda)$$
$$\mathbf{a} \longmapsto (P(\mathbf{a}), Q(\mathbf{a}))$$

In terms of characters, $h_{(1^k)} = \sum_{|\lambda|=k} K_{\lambda(1^k)} s_\lambda$

• When $n = k$ and restricted to $\{\mathbf{a} = a_1 \ldots a_k \,|\, a_i \neq a_j \, (i \neq j)\}$, we have a bijection (✍1)

$$\mathfrak{S}_k \longrightarrow \bigsqcup_{|\lambda|=k} \mathsf{ST}_k(\lambda) \times \mathsf{ST}_k(\lambda)$$

2.2 Littlewood-Richardson rule

- $\lambda, \mu \in \mathscr{P}$ with $\mu \subset \lambda$

$$\lambda/\mu = (5,4,1,1)/(3,1) \quad \longleftrightarrow \quad \begin{array}{ccccc} \circ & \circ & \circ & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet & \\ \bullet & & & & \\ \bullet & & & & \end{array}$$

(skew Young diagram)

- A Young tableau T of shape λ/μ

  $w(T)$: the word of T

$$T = \begin{array}{ccccc} \circ & \circ & \circ & 1 & 2 \\ \circ & 2 & 2 & 3 & \\ 1 & & & & \\ 4 & & & & \end{array} \qquad w(T) = 2132214$$

- $\lambda, \mu, \nu \in \mathscr{P}$ with $\mu \subset \lambda$ and $|\lambda| = |\mu| + |\nu|$

- A Young tableau T is called

a Littlewood-Richardson tableau of shape λ/μ with weight ν if

(1) the shape of $T = \lambda/\mu$

(2) the weight of $T = \nu$

(3) $w(T)$: a reverse lattice word

i.e. $w(T) = w_1 \dots w_k \quad \forall 1 \leq r \leq k$,

the $\sharp$ of i in $w_1 \dots w_r \geq$ the $\sharp$ of $i+1$ in $w_1 \dots w_r$ for all $i \geq 1$

e.g.

$$T = \begin{array}{ccccc} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & 1 & 2 & 2 & \\ 3 & & & & \\ 4 & & & & \end{array}$$

LR tableau of shape $(5,4,1,1)/(3,1)$ with content $(3,2,1,1)$

But

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & 1 & 2 \\ \bullet & 1 & 2 & 2 & \\ 3 & & & & \\ 4 & & & & \end{array} \qquad \begin{array}{ccccc} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & 2 & 2 & 2 & \\ 3 & & & & \\ 4 & & & & \end{array} \qquad \begin{array}{ccccc} \bullet & \bullet & \bullet & 1 & 1 \\ \bullet & 1 & 2 & 2 & \\ 3 & & & & \\ 5 & & & & \end{array} \qquad \times$$

- $\mathrm{LR}^{\lambda}_{\mu\nu}$: the set of LR tableau of shape λ/μ with content ν

- Theorem (Littlewood-Richardson rule)

We have a weight preserving bijection

$$\mathsf{SST}_n(\mu) \times \mathsf{SST}_n(\nu) \longrightarrow \bigsqcup_{\lambda} \mathsf{SST}_n(\lambda) \times \mathsf{LR}^{\lambda}_{\mu\nu}$$

Hence,

$$c^{\lambda}_{\mu\nu} = \left|\mathsf{LR}^{\lambda}_{\mu\nu}\right|$$

where $s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda$

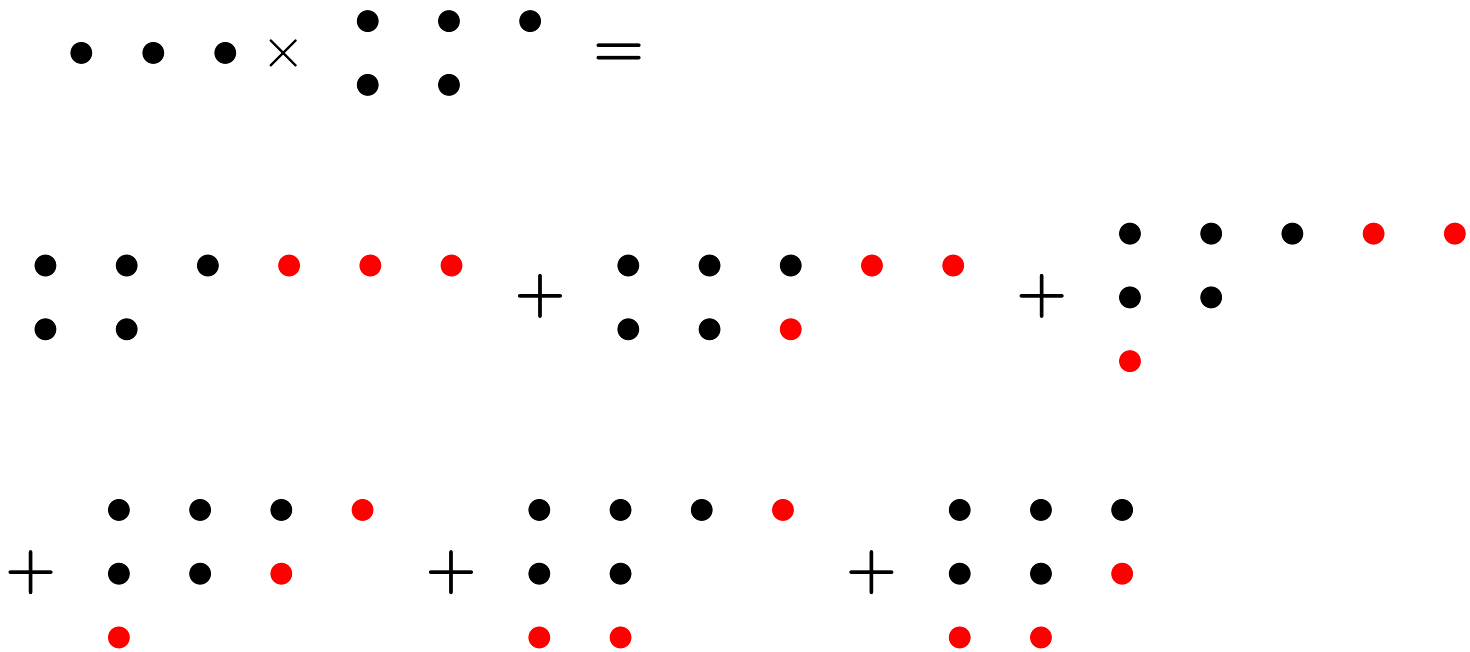
e.g. (1)

$$T = \begin{array}{lll} 1 & 2 & 2 \\ 3 & & \end{array} \qquad S = \begin{array}{ll} 2 & 2 \\ 3 & 4 \\ 5 & \end{array}$$

$$S = \begin{array}{ll} 2_1 & 2_1 \\ 3_2 & 4_2 \\ 5_3 & \end{array} \rightsquigarrow w = 2_1 2_1 4_2 3_2 5_3$$

$$P := (w \longrightarrow T) = \begin{array}{llll} 1 & 2 & 2 & 2 \\ 2 & 3 & & \\ 3 & 4 & & \\ 5 & & & \end{array} \qquad Q := \begin{array}{llll} \bullet & \bullet & \bullet & 1 \\ \bullet & 1 & & \\ 2 & 2 & & \\ 3 & & & \end{array}$$

(2)



In general,

$$h_r s_\mu = \sum_\lambda s_\lambda$$

where λ/μ is a horizontal strip of length r (Pieri's rule)

(3) $\mu = (\mu_1, \ldots, \mu_r > 0)$

The coefficient of s_λ in $h_\mu = h_{\mu_1} \cdots h_{\mu_r}$

= the $\sharp$ of chains of partitions

$$\emptyset = \nu_0 \subset \nu_1 \subset \cdots \subset \nu_r = \lambda$$

such that ν_k/ν_{k-1} : a horizontal strip of length μ_k $(1 \leq k \leq r)$

$= K_{\lambda\mu}$

$$h_\mu = s_\mu + \sum_\lambda K_{\lambda\mu} s_\lambda$$

(4) State and prove the analogues of (2) and (3) for elementary symmetric functions (✍2)

2.3 Knuth correspondence

For example,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \longleftrightarrow w = 3_1 1_1 2_2 2_2 1_2 2_3$$

we read the row indices of non-zero entries in each column from left to right, and from bottom to top in each column

$$P(A) := P(w) = \begin{matrix} 1 & 1 & 2 & 3 \\ 2 & 2 & & \end{matrix}, \qquad Q(A) := Q(w) = \begin{matrix} 1 & 1 & 2 & 2 \\ 2 & 3 & & \end{matrix}$$

- Theorem (Knuth)

(1) *We have a bijection*

$$\kappa : \mathsf{Mat}_{n\times n}(\mathbb{Z}_{\geq 0}) \longrightarrow \bigsqcup_{\lambda} \mathsf{SST}_n(\lambda) \times \mathsf{SST}_n(\lambda)$$
$$A \longmapsto (P(A), Q(A))$$

(2) $P(A^t) = Q(A)$

In particular, we can also recover

$$\kappa : \mathfrak{S}_n \longrightarrow \bigsqcup_{\lambda} \mathsf{ST}_n(\lambda) \times \mathsf{ST}_n(\lambda)$$

$$\kappa : \mathsf{SST}_n(\mu_1) \times \cdots \times \mathsf{SST}_n(\mu_r) \longrightarrow \bigsqcup_{\lambda} \mathsf{SST}_n(\lambda) \times \mathsf{SST}_n(\lambda)_\mu$$

- $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$

E_{ij} : elementary matrix $\longleftrightarrow x_i y_j$

$A \longleftrightarrow \prod_{1 \leq i,j \leq n} (x_i y_j)^{a_{ij}}$

$\mathrm{Mat}_{n \times n}(\mathbb{Z}_{\geq 0}) \longleftrightarrow \dfrac{1}{\prod_{1 \leq i,j \leq n}(1 - x_i y_j)}$

By taking $n \to \infty$, we recover the Cauchy identity

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

• Variations

We have bijections

(1)

$$\kappa : \mathsf{Mat}_{n\times n}(\mathbb{Z}_{\geq 0})^{\circ} \longrightarrow \bigsqcup_{\lambda} \mathsf{SST}_n(\lambda)$$
$$A \longmapsto P(A)$$

where $\mathsf{Mat}_{n\times n}(\mathbb{Z}_{\geq 0})^{\circ} = \{\, A \,|\, A = A^t \,\}$.

(2)

$$\kappa : \mathsf{Mat}_{n\times n}(\mathbb{Z}_2) \longrightarrow \bigsqcup_{\lambda} \mathsf{SST}_n(\lambda) \times \mathsf{SST}_n(\lambda')$$
$$A \longmapsto (P(A), Q(A))$$

where we read the entries of $A \in \mathsf{Mat}_{n\times n}(\mathbb{Z}_2)$ in a reverse way.

Find the identities corresponding to (1) and (2) (✍3).