Introduction to Symmetric functions and Representation theory

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2015.2

3. Representations of symmetric groups

- References

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Goal:

to understand and explain the representation theory of symmetric groups in terms of symmetric functions

3.1 Representations of a finite group

Suppose that our base field is $\mathbb{C}$

• G : a finite group

$\rho : G \longrightarrow GL(V, \mathbb{C})$: a complex fin. dim. representation of G

(We call V a G-module, and write $\rho(g)(v) = g \cdot v$ for $g \in G$, $v \in V$)

• The notion of homomorphism, submodule, simple (or irreducible) module are defined in a natural way.

- Examples

(1) $\mathbb{C}$ is a module over the cyclic group $\langle a \rangle$ of order n, where $a \cdot 1 = e^{\frac{2\pi i}{n}}$.

(2) $\mathbb{C}^n = \bigoplus_{i=1}^n \mathbb{C}e_i$ is an $\mathfrak{S}_n$-module, where $\sigma(e_i) = e_{\sigma(i)}$.

(3) $\mathbb{C}[G] = \bigoplus_{g \in G} \mathbb{C}v_g$: the group algebra of G with $v_g v_h = v_{gh}$

$\mathbb{C}[G]$ is a G-module where $g \cdot v_h = v_{gh}$ for $g, h \in G$

(regular representation of G)

Assume that every G-module is finite-dimensional

- Basic Results-1

(1) Every G-module V is completely reducible.

(V is a direct sum of simple G-modules)

(2) $\sharp$ of simple G-modules $=$ $\sharp$ of conjugacy classes in G

($\exists$ only finitely many irreducible G-modules, say $V_1, \ldots, V_r$).

e.g. $\sharp$ of simple $\mathfrak{S}_n$-modules $=$ $\sharp$ of partitions of n

- $\chi_V : G \longrightarrow \mathbb{C}$: the character of V

$$\chi_V(x) := \mathrm{tr}(\rho(x)|V) \qquad (x \in G)$$

(χ_V is a class function, i.e. $\chi_V(yxy^{-1}) = \chi_V(x)$)

- Class(G) : the space of all class functions on G

Class(G) has an inner product $\langle \ , \ \rangle_G$

$$\langle f, g \rangle_G := \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$$

- Basic Results-2

$\mathcal{B}_G = \{ \chi_{V_1}, \ldots, \chi_{V_r} \}$: an orthonormal basis of Class(G)

• V : a G-module

We can describe its decomposition into simple G-modules provided $\chi_{V_i} \in \mathcal{B}_G$ are known, that is,

$$V \cong V_1^{\oplus m_1} \oplus \cdots \oplus V_r^{\oplus m_r},$$

where

$$m_i = \langle \chi_V, \chi_{V_i} \rangle$$

for $1 \leq i \leq r$.

3.2 Irreducible characters for $\mathfrak{S}_n$

Goal Describe all irreducible characters of $\mathfrak{S}_n$

- $\mathcal{R}^n$: free abelian group generated by $\mathcal{B}_{\mathfrak{S}_n}$ for $n \geq 1$

$$\mathcal{R} := \bigoplus_{n \geq 0} \mathcal{R}^n$$

where $\mathcal{R}^0 := \mathbb{Z}$

- $\mathcal{R}^n_{\mathbb{C}} = \mathrm{Class}(\mathfrak{S}_n)$ and $\dim \mathcal{R}^n_{\mathbb{C}} = \sharp$ of partitions of n

• $\mathcal{R}$ is a graded Hopf algebra over $\mathbb{Z}$ with inner product.

for $f \in \mathcal{R}^m$, $g \in \mathcal{R}^n$,

$$f \cdot g := \mathrm{ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (f \times g) \quad \in \mathcal{R}^{m+n} \qquad \text{(mult.)}$$

$$\delta(f) := \sum_{p+q=m} \mathrm{res}_{\mathfrak{S}_p \times \mathfrak{S}_q}^{\mathfrak{S}_m} (f) \quad \in \mathcal{R} \otimes \mathcal{R} \qquad \text{(comult.)}$$

for $f = \sum f_n$, $g = \sum g_n \in \mathcal{R}$,

$$\langle f, g \rangle := \sum_{n \geq 0} \langle f_n, g_n \rangle_{\mathfrak{S}_n} \qquad \text{(inn. prod.)}$$

• Characteristic map $\mathrm{ch} : \mathcal{R} \longrightarrow \Lambda_{\mathbb{C}}$

$$\mathrm{ch}(f) := \sum_{|\mu|=n} \frac{1}{z_\mu} f(\mu) p_\mu \qquad (f \in \mathcal{R}^n)$$

e.g. (1) f : the trivial repn $\mathbb{C}$ of $\mathfrak{S}_n$

$$\mathrm{ch}(f) = \sum_{|\mu|=n} \frac{1}{z_\mu} p_\mu = h_n$$

(2) f : the regular repn $\mathbb{C}[\mathfrak{S}_n]$ of $\mathfrak{S}_n$

$$\mathrm{ch}(f) = p_{(1,\ldots,1)} = p_1^n = h_1^n$$

(3) $\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$ $\quad f = \mathrm{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\mathbf{1}_\lambda) = \mathbf{1}_{\lambda_1} \cdots \mathbf{1}_{\lambda_r}$,

$$\mathrm{ch}(f) = h_{\lambda_1} \cdots h_{\lambda_r} = h_\lambda$$

- Theorem

  ch : $\mathcal{R} \longrightarrow \Lambda$ *is an isometric iso. of graded Hopf alg. with*

$$\mathrm{ch}(\mathcal{B}_{\mathfrak{S}_n}) = \{\, s_\lambda \mid |\lambda| = n \,\}$$

- $\chi^\lambda = \mathrm{ch}^{-1}(s_\lambda)$: irreducible character corresponding to λ

  (Frobenius formula)

$$s_\lambda = \sum_{|\mu|=n} \frac{1}{z_\mu} \chi^\lambda(\mu) p_\mu \quad \text{or} \quad p_\mu = \sum_{|\mu|=n} \chi^\lambda(\mu) s_\lambda$$

※ *The character table* $\left[\chi^\lambda(\mu)\right]_{\lambda,\mu}$ *can be obtained from the transition matrix between* $\{\, s_\lambda \,\}$ *and* $\{\, p_\mu \,\}$ *(in* $\Lambda_{\mathbb{Q}}$*)*

• (Murnaghan-Nakayama rule)

There exists a combinatorial formula for $\chi^\lambda(\mu)$, which can be obtained by using the Frobenius formula

$$\chi^\lambda(\mu) = \sum_T (-1)^{\mathrm{ht}(T)} \in \mathbb{Z}$$

where the sum is over the rim hook tableaux of shape λ and content μ.

e.g.

$$T \quad = \quad \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & \\ 1 & 3 & & \\ 3 & 3 & & \end{matrix}$$

of shape $\lambda = (4, 3, 2, 2)$ and content $\mu = (4, 4, 3)$

- In particular, $\chi^\lambda(1^n) = |\mathsf{ST}_n(\lambda)|$

- (Young)

There exists an explicit construction of irreducible $\mathfrak{S}_n$-module V^λ with character χ^λ and linear basis $\mathsf{ST}_n(\lambda)$.

- Transition matrices and multiplicities of irreducible repn's

(1) $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $|\lambda| = n$, $\quad \mathfrak{S}_\lambda = \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$

$$f = \mathrm{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\mathbf{1}_\lambda) = \mathbf{1}_{\lambda_1} \cdots \mathbf{1}_{\lambda_r}, \qquad \mathrm{ch}(f) = h_{\lambda_1} \cdots h_{\lambda_r} = h_\lambda$$

$$h_\lambda = s_\lambda + \sum_\mu K_{\mu\lambda} s_\mu \quad (\because s_\lambda = m_\lambda + \sum_{\lambda > \mu} K_{\lambda\,\mu} m_\mu)$$

$$\therefore \quad f = \chi^\lambda + \sum_{\mu > \lambda} K_{\mu\lambda} \chi^\mu$$

(2) In particular, $\lambda = (1, \cdots, 1) = (1^n)$ $\quad$ (i.e. $f = \chi_{\mathbb{C}[\mathfrak{S}_n]}$)

$$h_{(1^n)} = \sum_\mu K_{\mu(1^n)} s_\mu$$

$$K_{\mu(1^n)} = |\mathrm{ST}_\mathrm{n}(\mu)| = \chi^\mu(1^n)$$

Proof of Theorem (Brief sketch)

(1) ch preserves mult. and inner products.

(2) Put $\chi^\lambda = \det(\eta_{\lambda_i - i + j})_{1 \le i,j \le n}$, where $\eta_\lambda = \mathrm{ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\mathbf{1}_\lambda)$. Then

$$\mathrm{ch}(\chi^\lambda) = s_\lambda$$

(3) $\{\, \chi^\lambda \mid \lambda \in \mathscr{P} \,\}$: a basis of $\mathcal{R}$, and ch : a ring isomorphism

(4) $\chi^\lambda(1) = \langle s_\lambda, p_{(1^n)} \rangle = \langle s_\lambda, h_{(1^n)} \rangle = K_{\lambda(1^n)} > 0$

(5) δ is equal to the induced comult. by ch