Preliminaries: Coloring and List Coloring of Graphs

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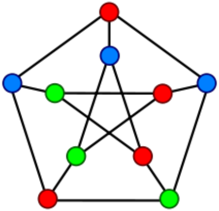
Enumerative Combinatorics School 2014

Coloring



Definition

- A (proper) coloring of a graph G is a labelling of the vertices of G with colors so that no two adjacent vertices have the same color.
- A k-coloring is a coloring using at most k colors.
- The chromatic number $\chi(G)$ of G is the smallest integer k such that G admits k-coloring.

Example

Two Famous Results of Graph Coloring at Textbook

Brook's Theorem (Brooks, 1941)

Vizing's Theorem (Vizing, 1964)

List Coloring

Definition

- A list assignment L of a graph G is a function on $V(G)$ such that $\forall v \in V(G)$, $L(v)$ is a finite nonempty set.
- For a list assignment L of a graph G, G is L-choosable if there is a coloring c of G such that $\forall v \in V(G)$, $c(v) \in L(v)$.

Example

- A graph G is k-choosable if G is L-choosable for any list assignment L such that $v \in V(G)$, $|L(v)| \geq k$.
- The list chromatic number $\chi_\ell(G)$ of a graph G is the smallest integer k such that G is k-choosable.

List Coloring

In general, $\chi(G) \leq \chi_\ell(G)$.

Example $\chi(K_{2\star n}) = \chi_\ell(K_{2\star n})$

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Example $\chi_\ell(K_{\binom{n}{k}, \lfloor \frac{n}{k} \rfloor}) > k$

Theorem (Alon, 1992)

$\chi_\ell(K_{m,n}) = \Theta(n \log m)$

Theorem (Kierstead, 2000)

$\chi_\ell(K_{3\star n}) = \left\lceil \frac{4n-1}{3} \right\rceil$

Theorem (Yang 2003 / Noel, West, Wu, Zhu, 2014)

$\left\lfloor \frac{3n}{2} \right\rfloor \leq \chi_\ell(K_{4\star n}) = \left\lceil \frac{5n-1}{3} \right\rceil$

List Version of Brook's Theorem and Vizing's Theorem

List Version of Brook's Theorem?

List Version of Vizing's Theorem?

Online List Coloring (Paintability)

Online List Coloring

Given a graph G and a function $f\colon V(G) \to \mathbb{N}$, Alice and Bob play the following game.

- At the initial step, every vertex is a non-finished vertex.
- In the ith step, Alice chooses a non-empty subset V_i of non-finished vertices among $V(G)$ and Bob chooses an independent set X_i such that $X_i \subset V_i$.
  If a vertex v is contained in $f(v)$ of V_j's, then v finished.
- The game terminates if all the vertices are finished.
- Bob wins if the union of all X_i is $V(G)$.

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- Given a graph G and a function $f: V(G) \to \mathbb{N}$,
  G is online f-choosable if Bob always has a winning strategy.
- A graph G is online k-choosable if G is online f-choosable for any function f such that $f(v) \geq k$ for every $v \in V(G)$.
- The online list chromatic number $\chi^{ol}(G)$ is the smallest integer k such that G is online k-choosable.

Questions on online list coloring

- We do not know $\chi^{ol}(G)$ for almost all graphs.
- Does there exist G such that $\chi^{ol}(G) - \chi_\ell(G)$ is arbitrarily large?
- Is a graph G online chromatic-choosable if $|V(G)| \leq 2\chi(G)$?

From now on, we will see one algebraic method for graph choosability.