Graph Coloring and Orientations

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Alon and Tarsi's Theorem

Definition

Let D be a digraph D and H be a subdigraph of D.

- H is Eulerian if for any $v \in V(D)$, $d_H^+(v) = d_H^-(v)$.

Example

Alon and Tarsi's Theorem

Alon and Tarsi's Theorem (Alon and Tarsi, 1992)

Example

Proof of Alon and Tarsi's Theorem

Proof of Alon and Tarsi's Theorem

Lemma

Let $P := P(x_1, \ldots, x_n)$ be a polynomial over $\mathbb{Z}$.
Let $S_i \subset \mathbb{Z}$ such that $|S_i| = \deg_{x_i}(P) + 1$ for each $1 \leq i \leq n$.
If $P(s_1, \ldots, s_n) = 0$ for all $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ then $P \equiv 0$.

Sketch of Proof

Graph polynomial

Graph polynomial

Proof of Alon and Tarsi's Theorem

Lemma

Let D be an orientation of G, let $d_i = d_D^+(x_i)$ for each i. Then

$$|DE_G(d_1, d_2, \ldots, d_n) - DO_G(d_1, d_2, \ldots, d_n)| =$$

Sketch of Proof

Proof of Alon and Tarsi's Theorem

Alon and Tarsi's Theorem (Alon and Tarsi, 1992)

Corollaries

If D is an acyclic orientation of G, then G is $(\triangle^+(D)+1)$-choosable.

Let
$$A(G) = \max\left\{\frac{|E(H)|}{|V(H)|} \mid H: \text{ subgraph of } G\right\}$$

- (Tarsi, 1981) A graph G has an orientation of maximum out-degree is at most d if and only if $A(G) \leq d$.

-Every bipartite graph G is $(A(G)+1)$-choosable.

-Every planar bipartite graph is 3-choosable.

Algebraic proof of Brook's Theorem

Algebraic proof of Brook's Theorem

It is easy to see that list version of Brooks's Theorem is true for Gallai trees.

- A Gallai tree is a connected graph whose blocks are either a complete graph or an odd cycle.

Lemma

A connected graph G which is not a Gallai tree contains an even cycle with at most on chord as an induced subgraph.

Algebraic proof of Brook's Theorem

Lemma

A connected graph G which is not a Gallai tree contains an even cycle with at most on chord as an induced subgraph.

Sketch of Proof

1. G has a cycle C such that $G[C]$ is neither a complete graph nor an odd cycle.

2. Take such C with the minimum length ℓ and suppose that $\ell \geq 5$.

3. There is no vertex $x \in C$ such that $v \sim w$ for all $w \in V(C)$.

4. Each vertex is incident to at most one chord.

5. C contains at most one chord.

Algebraic proof of Brook's Theorem

List Version of Brook's Theorem

Sketch of Proof

List Coloring Conjecture

List Coloring Conjecture

If $\chi_\ell(G) = \chi(G)$, then G is called chromatic-choosable.

List (edge) Coloring Conjecture

Total Coloring Conjecture

List Coloring Conjecture

Theorem [Galvin, 1995]

For every bipartite G, $\chi_l(L(G)) = \chi(L(G))$.

Theorem [Haggkvist and Janssen, 1997]

For any integer n, $\chi_l(L(K_n)) \leq n$.

(This implies that LCC holds for K_n when n is odd.)

Theorem [Borodin, Kostochka, Woodall, 2001]

If G is a planar graph with $\Delta \geq 12$, $\chi_l(L(G)) = \chi(L(G))$.

Using orientation, but not Alon-Tarsi's Thereom

A kernel S of a digraph D is an independent set of D such that for any $v \in V(D) \setminus S$, there is $w \in S$ such that $(v, w) \in A(D)$.

Proposition

Let D be an orientation of G. Suppose that every induced subdigraph of D has a kernel.
Then G is $(\triangle^+(D) + 1)$-choosable.

Example For every bipartite G, $\chi_l(L(G)) = \chi(L(G))$ (Galvin, 1995)

The converse is...?

Alon-Tarsi number

Online choosability Version of Alon-Tarsi's Theorem

Online choosability Version of Alon-Tarsi's Theorem

Theorem

If one monomial $\Pi_{v \in V(G)} x_v^{\varphi(v)}$ of P_G has nonzero coefficient with $\varphi(v) < k$ then G is online $(k+1)$-choosable.

Example $K_{2\star k}$ is online k-choosable.

Hypergraph Extension of Alon and Tarsi's Theorem

Coloring of a hypergraph and hypergraph polynomials

- A (proper) coloring of a hypergraph H is a labelling of the vertices of H with colors so that there is no mono-colored hyperedge.

Hypergraph polynomials for k-uniform hypergpraph

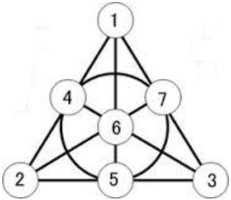
Orientations of a hypergraph

Let H be a hypergraph with $V(H) = \{x_1, x_2, \ldots, x_n\}$

- An orientation D of H is choice of a source vertex from each hyperedge in H.
- $d_D^+(x_i)$: the number of hyperedges with source x_i in orientation D
- The weight of a hyperedge and the weight of an orientation are defined by
- The modularity $m(D)$ is $\sum_{i=0}^{k-1} im_i$ where m_i is the number of edges with source in ith position.

Orientations of a hypergraph

Example Fano plane F



Expanding f_H

Balanced partition

Let D be an orientation of a k-uniform hypergraph H.
Let $\mathcal{B} = (B_0, B_1, \dots, B_{k-1})$ be a partition of $E(H)$.

- The modular sum of $\mathcal{B}$ is

$$m(\mathcal{B}) = \sum_{i=0}^{k-1} i \times |B_i| \pmod{k}.$$

- For each vertex v, for each B_j, $d^i_{B_j}(v)$ be the number of edges in B_j such that v appears i positions after the position of the source of the edges, cyclically.
- $\mathcal{B}$ is a balanced partition if for all vertex v,

$$\sum_{i=0}^{k-1} d^0_{B_i} = \sum_{i=0}^{k-1} d^i_{B_i}(v).$$

Theorem

Let D be an orientation of a k-uniform hypergraph H.
Suppose that the number of balanced partitions with modular sum i is not dependent of i.
Then H is $(\triangle^+(D) + 1)$-choosable.